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2004 J. Phys. A: Math. Gen. 37 433

(http://iopscience.iop.org/0305-4470/37/2/012)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 433–440 PII: S0305-4470(04)62066-X

Bethe ansatz solution of the open XXZ chain with nondiagonal boundary terms

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Received 11 April 2003
Published 15 December 2003
Online at stacks.iop.org/JPhysA/37/433 (DOI: 10.1088/0305-4470/37/2/012)

Abstract
We propose a set of conventional Bethe ansatz equations and a corresponding expression for the eigenvalues of the transfer matrix for the open spin-$\frac{1}{2}$ XXZ quantum spin chain with nondiagonal boundary terms, provided that the boundary parameters obey a certain linear relation.

PACS numbers: 75.10.Jm, 02.30.Ik

1. Introduction

Consider the open spin-$\frac{1}{2}$ XXZ quantum spin chain with nondiagonal boundary terms, defined by the Hamiltonian [1]

$$\mathcal{H} = \frac{1}{2}\left\{\sum_{n=1}^{N-1}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \cosh\eta\,\sigma_n^z\sigma_{n+1}^z\right) + \sinh\eta\left[\coth\xi_-\sigma_1^z + \frac{2\kappa_-}{\sinh\xi_-}\left(\cosh\theta_-\sigma_1^x + \mathrm{i}\sinh\theta_-\sigma_1^y\right) - \coth\xi_+\sigma_N^z - \frac{2\kappa_+}{\sinh\xi_+}\left(\cosh\theta_+\sigma_N^x + \mathrm{i}\sinh\theta_+\sigma_N^y\right)\right]\right\} \qquad (1.1)$$

where $\sigma^x, \sigma^y, \sigma^z$ are the usual Pauli matrices, η is the bulk anisotropy parameter, $\xi_\pm, \kappa_\pm, \theta_\pm$ are arbitrary boundary parameters and N is the number of spins. This is the prototypical integrable quantum spin chain with boundary. It is related to many other models, including the sine-Gordon field theory [2]. Moreover, this model has applications in various branches of physics, including condensed matter and statistical mechanics.

This model has resisted solution for many years (see, e.g., [3]). The main difficulty is that, in contrast to the special case of diagonal boundary terms (i.e., $\kappa_\pm = 0$) considered in [4, 5], a simple pseudovacuum (reference) state does not exist. For example, the state with all spins up $\binom{1}{0}^{\otimes N}$ is not an eigenstate of the Hamiltonian.

We recently formulated [6] a method of deriving the Bethe ansatz solution of integrable spin chain (vertex-type) models which does not rely on the existence of a pseudovacuum state. In particular, we used this method to solve the model (1.1) for the special case

$$\kappa_+ = \kappa_- \qquad \xi_+ = \xi_- \qquad \theta_+ = \theta_- = 0 \qquad N = \text{odd}. \tag{1.2}$$

Here we propose the solution for a more general case. Indeed, in terms of the boundary parameters $\alpha_\mp$, $\beta_\mp$ introduced in equation (3.24), we find an expression for the eigenvalues of the transfer matrix corresponding to the Hamiltonian

$$\begin{aligned}\mathcal{H} = \frac{1}{2}\Bigg\{&\sum_{n=1}^{N-1}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \cosh\eta\,\sigma_n^z\sigma_{n+1}^z\right)\\ &+ \sinh\eta\big[\coth\alpha_-\tanh\beta_-\sigma_1^z + \operatorname{csch}\alpha_-\operatorname{sech}\beta_-\left(\cosh\theta_-\sigma_1^x + \mathrm{i}\sinh\theta_-\sigma_1^y\right)\\ &- \coth\alpha_+\tanh\beta_+\sigma_N^z + \operatorname{csch}\alpha_+\operatorname{sech}\beta_+\left(\cosh\theta_+\sigma_N^x + \mathrm{i}\sinh\theta_+\sigma_N^y\right)\big]\Bigg\}\end{aligned} \tag{1.3}$$

where the boundary parameters are subject to the linear relation

$$\alpha_- + \beta_- + \alpha_+ + \beta_+ = \pm(\theta_- - \theta_+) + \eta k \tag{1.4}$$

where k is an even integer if N is odd, and is an odd integer if N is even. In the recent paper [7], similar results have been obtained by a different approach.

The outline of this paper is as follows. In section 2, we briefly review the construction of the model's transfer matrix, and list some of its important properties. In section 3, we find the eigenvalues of the transfer matrix by the three-step procedure formulated in [6]. The first two steps, which lead to a functional relation for the transfer matrix, are the same as in [6, 8], except for the introduction of the parameters $\theta_\mp$. The principal new results appear at the third step, where we succeed to recast the functional relation in terms of a determinant for the more general case (1.4). We conclude with a brief discussion of our results in section 4.

2. The transfer matrix

The fundamental transfer matrix $t(u)$ corresponding to the model (1.1) is given by [5]

$$t(u) = \operatorname{tr}_0 K_0^+(u)T_0(u)K_0^-(u)\hat{T}_0(u) \tag{2.1}$$

where the monodromy matrices are given by

$$T_0(u) = R_{0N}(u)\cdots R_{01}(u) \qquad \hat{T}_0(u) = R_{10}(u)\cdots R_{N0}(u) \tag{2.2}$$

and the R matrix is the solution of the Yang–Baxter equation given by

$$R(u) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0\\ 0 & \sinh u & \sinh\eta & 0\\ 0 & \sinh\eta & \sinh u & 0\\ 0 & 0 & 0 & \sinh(u+\eta)\end{pmatrix}. \tag{2.3}$$

Moreover, the K^- matrix is the solution of the boundary Yang–Baxter equation [9] given by [1, 2]

$$K^-(u) = \begin{pmatrix} \sinh(\xi_- + u) & \kappa_- \mathrm{e}^{\theta_-}\sinh 2u\\ \kappa_-\mathrm{e}^{-\theta_-}\sinh 2u & \sinh(\xi_- - u)\end{pmatrix} \tag{2.4}$$

which evidently depends on three boundary parameters $\xi_-, \kappa_-, \theta_-$. It is related to the symmetric matrix $K^-(u)|_{\theta_-=0}$ used in [6, 8] by a gauge transformation,

$$K^-(u) = \mathcal{M}\,K^-(u)|_{\theta_-=0}\,\mathcal{M}^{-1} \tag{2.5}$$

with

$$\mathcal{M} = \begin{pmatrix} e^{\frac{1}{2}\theta_-} & 0 \\ 0 & e^{-\frac{1}{2}\theta_-} \end{pmatrix}. \tag{2.6}$$

The matrix $K^+(u)$ is equal to $K^-(-u-\eta)$ with $(\xi_-, \kappa_-, \theta_-)$ replaced by $(\xi_+, \kappa_+, \theta_+)$. Finally, tr_0 denotes trace over the (two-dimensional) 'auxiliary space' 0. Further details about the construction of this transfer matrix can be found in [5, 8].

The transfer matrix constitutes a one-parameter commutative family of matrices

$$[t(u), t(v)] = 0. \tag{2.7}$$

The Hamiltonian (1.1) is related to the first derivative of the transfer matrix,

$$\mathcal{H} = c_1 \frac{\partial}{\partial u} t(u)\Big|_{u=0} + c_2 \mathbb{I} \tag{2.8}$$

where

$$c_1 = \frac{1}{4\sinh\xi_- \sinh\xi_+ \sinh^{2N-1}\eta \cosh\eta} \qquad c_2 = -\frac{\sinh^2\eta + N\cosh^2\eta}{2\cosh\eta} \tag{2.9}$$

and $\mathbb{I}$ is the identity matrix. The two relations (2.7), (2.8) signal that the model is integrable. Moreover, it is evident that in order to determine the energy eigenvalues, it suffices to determine the eigenvalues of the transfer matrix.

The transfer matrix has the periodicity property

$$t(u + i\pi) = t(u) \tag{2.10}$$

as well as crossing symmetry

$$t(-u-\eta) = t(u) \tag{2.11}$$

and the asymptotic behaviour (for $\kappa_\pm \neq 0$)

$$t(u) \sim -\kappa_-\kappa_+ \cosh(\theta_- - \theta_+) \frac{e^{u(2N+4)+\eta(N+2)}}{2^{2N+1}} \mathbb{I} + \cdots \qquad \text{for} \quad u \to \infty. \tag{2.12}$$

3. Bethe ansatz solution

We now proceed to find an expression for the transfer matrix eigenvalues using the method formulated in [6]. This method consists of three main steps.

3.1. Step 1: fusion hierarchy

The first step is to obtain the model's so-called fusion hierarchy. The transfer matrix (2.1) is actually the first $\left(j = \frac{1}{2}\right)$ member of an infinite hierarchy of commuting transfer matrices $t^{(j)}(u)$ corresponding to spin-j (i.e., $(2j+1)$-dimensional) auxiliary spaces, $j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Using the fusion procedure for R [10, 11] and K [12, 13] matrices, one finds that these higher-level transfer matrices obey the relations

$$t^{(j)}(u) = \tilde{\zeta}_{2j-1}(2u + (2j-1)\eta) \Bigg[t^{(j-\frac{1}{2})}(u) t^{(\frac{1}{2})}(u + (2j-1)\eta) - \frac{\Delta(u + (2j-2)\eta)\tilde{\zeta}_{2j-2}(2u + (2j-2)\eta)}{\zeta(2u + 2(2j-1)\eta)} t^{(j-1)}(u) \Bigg] \tag{3.1}$$

with $t^{(0)} = \mathbb{I}$, and $j = 1, \frac{3}{2}, \ldots$. The quantum determinant $\Delta(u)$ is given by

$$\Delta(u) = -[\sinh(u+\eta+\xi_-)\sinh(u+\eta-\xi_-) + \kappa_-^2 \sinh^2(2u+2\eta)] \\ \times [\sinh(u+\eta+\xi_+)\sinh(u+\eta-\xi_+) + \kappa_+^2 \sinh^2(2u+2\eta)] \\ \times \sinh 2u \sinh(2u+4\eta)\zeta(u+\eta)^{2N} \tag{3.2}$$

and

$$\tilde{\zeta}_j(u) = \prod_{k=1}^{j} \zeta(u+k\eta) \qquad \tilde{\zeta}_0(u) = 1 \tag{3.3}$$

$$\zeta(u) = -\sinh(u+\eta)\sinh(u-\eta). \tag{3.4}$$

These relations are the same as those for the case of symmetric K matrices ($\theta_\mp = 0$) [8]. We remark that the spin-j matrix $K^-_{\langle 1\ldots 2j\rangle}(u)$ is related to the corresponding matrix with $\theta_- = 0$ by a generalization of the gauge transformation (2.5),

$$K^-_{\langle 1\ldots 2j\rangle}(u) = \mathcal{M}_1 \ldots \mathcal{M}_{2j}\, K^-_{\langle 1\ldots 2j\rangle}(u)|_{\theta_-=0}\, \mathcal{M}_{2j}^{-1} \ldots \mathcal{M}_1^{-1}. \tag{3.5}$$

3.2. Step 2: truncation at roots of unity

The second step is to observe [8] that for anisotropy values

$$\eta = \frac{\mathrm{i}\pi}{p+1} \qquad p = 1, 2, \ldots \tag{3.6}$$

(and hence $q \equiv \mathrm{e}^\eta$ is a root of unity, satisfying $q^{p+1} = -1$), the level-$\frac{p+1}{2}$ transfer matrix can be expressed in terms of a transfer matrix of one level lower,

$$t^{(\frac{p+1}{2})}(u) = \alpha(u)\big[t^{(\frac{p-1}{2})}(u+\eta) + \beta(u)\mathbb{I}\big]. \tag{3.7}$$

The quantities $\alpha(u)$ and $\beta(u)$ are given by the corresponding expressions (4.31) in [8], except $\sigma_\mp(u) \to \mathrm{e}^{(p+1)\theta_\mp}\sigma_\mp(u)$ and $\rho_\mp(u) \to \mathrm{e}^{-(p+1)\theta_\mp}\rho_\mp(u)$, as a consequence of (3.5).

This result provides an example of McCoy's dictum 'complicated is simple' [14]. Indeed, the essential point of this step is to exploit the higher symmetry which occurs at roots of unity to help solve the model.

Combining the fusion hierarchy (3.1) and the truncation identity (or 'closing relation') (3.7) for the η values (3.6), we arrive at a functional relation for the fundamental transfer matrix $t(u) \equiv t^{(\frac{1}{2})}(u)$ (and hence, for the corresponding eigenvalues $\Lambda(u)$) of order $p+1$ [6, 8]:

$$\Lambda(u)\Lambda(u+\eta)\ldots\Lambda(u+p\eta) - \delta(u-\eta)\Lambda(u+\eta)\Lambda(u+2\eta)\ldots\Lambda(u+(p-1)\eta) \\ - \delta(u)\Lambda(u+2\eta)\Lambda(u+3\eta)\ldots\Lambda(u+p\eta) \\ - \delta(u+\eta)\Lambda(u)\Lambda(u+3\eta)\Lambda(u+4\eta)\ldots\Lambda(u+p\eta) \\ - \delta(u+2\eta)\Lambda(u)\Lambda(u+\eta)\Lambda(u+4\eta)\cdots\Lambda(u+p\eta) - \cdots \\ - \delta(u+(p-1)\eta)\Lambda(u)\Lambda(u+\eta)\cdots\Lambda(u+(p-2)\eta) + \cdots = f(u) \tag{3.8}$$

where $\delta(u)$ is defined by

$$\delta(u) = \frac{\Delta(u)}{\zeta(2u+2\eta)}. \tag{3.9}$$

Moreover, the function $f(u)$ is given by

$$f(u) = \frac{(-1)^{p(N+1)}}{2^{2p(N+1)}} \sinh^{2N}((p+1)u) \frac{\cosh^2\left((p+1)u + \frac{\mathrm{i}\pi}{2}\epsilon\right)}{\cosh^2((p+1)u)} \\ \times \{n(u;\xi_-,\kappa_-)n(u;-\xi_+,\kappa_+) + n(u;-\xi_-,\kappa_-)n(u;\xi_+,\kappa_+) \\ + 2(-1)^N(-\kappa_-\kappa_+)^{p+1}\sinh^2(2(p+1)u)\cosh((p+1)(\theta_- - \theta_+))\} \tag{3.10}$$

where $\epsilon = 2\text{frac}(p/2)$ equals 0 if p is even, and equals 1 if p is odd; and the function $n(u;\xi,\kappa)$ is defined by

$$n(u;\xi,\kappa) = \sinh((p+1)(\xi+u)) + \sum_{l=1}^{[\frac{p+1}{2}]} c_{p,l}\kappa^{2l}\sinh((p+1)u + (p+1-2l)\xi) \tag{3.11}$$

with

$$c_{p,l} = \frac{(p+1)}{l!}\prod_{k=0}^{l-2}(p-l-k).$$

For instance, for the case $p = 3$, the functional relation is given by

$$\begin{aligned}\Lambda(u)\Lambda(u+\eta)\Lambda(u+2\eta)\Lambda(u+3\eta) - \delta(u-\eta)\Lambda(u+\eta)\Lambda(u+2\eta)\\ - \delta(u)\Lambda(u+2\eta)\Lambda(u+3\eta) - \delta(u+\eta)\Lambda(u)\Lambda(u+3\eta)\\ - \delta(u+2\eta)\Lambda(u)\Lambda(u+\eta) + \delta(u)\delta(u+2\eta)\\ + \delta(u-\eta)\delta(u+\eta) = f(u).\end{aligned} \tag{3.12}$$

3.3. Step 3: determinant representation

Following the strategy used in [15] to solve RSOS models, the third and final step is to rewrite the functional relation as the determinant of a $(p+1)\times(p+1)$ matrix. Let us assume that this matrix has the same form as the one for the diagonal case ($\kappa_\pm = 0$) and for the case (1.2). That is, we assume the functional relation can be cast in the form [6]

$$\det\begin{pmatrix} \Lambda_0 & -h'_{-1} & 0 & 0 & \dots & 0 & 0 & -h_0 \\ -h_1 & \Lambda_1 & -h'_0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -h_2 & \Lambda_2 & -h'_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -h_{p-1} & \Lambda_{p-1} & -h'_{p-2} \\ -h'_{p-1} & 0 & 0 & 0 & \dots & 0 & -h_p & \Lambda_p \end{pmatrix} = 0 \tag{3.13}$$

where $\Lambda_k = \Lambda(u+\eta k)$, $h_k = h(u+\eta k)$, $h'_k = h'(u+\eta k)$,

$$h'(u) = h(-u-2\eta) \tag{3.14}$$

and the function $h(u)$ is yet to be determined. We find that the functional relation (3.8) can indeed be recast in the form (3.13), provided that $h(u)$ satisfies the three conditions

$$h(u+\mathrm{i}\pi) = h(u) \tag{3.15}$$

$$h(u+\eta)h(-u-\eta) = \delta(u) \tag{3.16}$$

$$\prod_{j=0}^{p} h(u+j\eta) + \prod_{j=0}^{p} h(-u-j\eta) = f(u). \tag{3.17}$$

The results [6] for $h(u)$ in the diagonal case and in the case (1.2) suggest that, in general, $h(u)$ has form

$$h(u) = -\sinh^{2N}(u+\eta)\frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)}g_-(u)g_+(u) \tag{3.18}$$

where the functions $g_\mp(u)$ contain all the dependence on the boundary parameters.

Then second condition (3.16) together with (3.2), (3.9) and (3.18) implies that

$$g_-(u)g_+(u)g_-(-u)g_+(-u) = (\sinh^2 u - \sinh^2 \xi_- + \kappa_-^2 \sinh^2 2u) \times \left(\sinh^2 u - \sinh^2 \xi_+ + \kappa_+^2 \sinh^2 2u\right). \tag{3.19}$$

This suggests that $g_\mp(u)$ obey the functional equation

$$g_\mp(u)g_\mp(-u) = -\left(\sinh^2 u - \sinh^2 \xi_\mp + \kappa_\mp^2 \sinh^2 2u\right). \tag{3.20}$$

Assuming that the functions $g_\mp(u)$ are given by

$$g_\mp(u) = 2\kappa_\mp \sinh(u + \alpha_\mp)\cosh(u + \beta_\mp) \tag{3.21}$$

(3.20) implies that the parameters $\alpha_\mp$, $\beta_\mp$ obey

$$\sinh^2 \alpha_\mp \cosh^2 \beta_\mp = \frac{1}{4\kappa_\mp^2}\sinh^2 \xi_\mp \qquad \cosh^2 \alpha_\mp \sinh^2 \beta_\mp = \frac{1}{4\kappa_\mp^2}\cosh^2 \xi_\mp. \tag{3.22}$$

A similar reparametrization appears in [2, 7]. Below we shall argue that (3.21) is essentially the unique solution of (3.19).

The third condition (3.17) together with (3.10) and (3.18) implies that[1]

$$\begin{aligned}\prod_{j=0}^{p} g_-(u + j\eta)g_+(u + j\eta) + \prod_{j=0}^{p} g_-(-u - j\eta)g_+(-u - j\eta) \\ = \frac{(-1)^p}{2^{2p}}\{n(u; \xi_-, \kappa_-)n(u; -\xi_+, \kappa_+) + n(u; -\xi_-, \kappa_-)n(u; \xi_+, \kappa_+) \\ + 2(-1)^N(-\kappa_-\kappa_+)^{p+1}\sinh^2(2(p + 1)u)\cosh((p + 1)(\theta_- - \theta_+))\}\end{aligned} \tag{3.23}$$

where $n(u; \xi, \kappa)$ is given by equation (3.11). We find that this requirement can be satisfied for p = odd, with $g_\mp(u)$ given by (3.21) and[2]

$$\begin{aligned}\sinh\alpha_- \cosh\beta_- &= \frac{1}{2\kappa_-}\sinh\xi_- \qquad & \cosh\alpha_- \sinh\beta_- &= \frac{1}{2\kappa_-}\cosh\xi_- \\ \sinh\alpha_+ \cosh\beta_+ &= -\frac{1}{2\kappa_+}\sinh\xi_+ \qquad & \cosh\alpha_+ \sinh\beta_+ &= -\frac{1}{2\kappa_+}\cosh\xi_+\end{aligned} \tag{3.24}$$

provided that the various parameters obey the linear constraint

$$\alpha_- + \beta_- + \alpha_+ + \beta_+ = \pm(\theta_- - \theta_+) + \eta k \tag{3.25}$$

where k is an even integer if N is odd, and is an odd integer if N is even.

In short, the functional relations can be cast in the determinant form (3.13) for p = odd with

$$h(u) = -\sinh^{2N}(u + \eta)\frac{\sinh(2u + 2\eta)}{\sinh(2u + \eta)} \times 4\kappa_-\kappa_+ \sinh(u + \alpha_-)\cosh(u + \beta_-)\sinh(u + \alpha_+)\cosh(u + \beta_+) \tag{3.26}$$

where $\alpha_\mp$, $\beta_\mp$ are defined by (3.24) and satisfy the constraint (3.25).

We now proceed as in [6, 15], and assume that the matrix in (3.13) has the null vector $(Q_0, Q_1, \ldots, Q_p)$. That is,

$$\begin{aligned}&\Lambda_0 Q_0 - h'_{-1} Q_1 - h_0 Q_p = 0 \\ &-h_k Q_{k-1} + \Lambda_k Q_k - h'_{k-1} Q_{k+1} = 0 \qquad k = 1, \ldots, p - 1 \\ &-h'_{p-1} Q_0 - h_p Q_{p-1} + \Lambda_p Q_p = 0.\end{aligned} \tag{3.27}$$

[1] Note that the right-hand side of (3.23) depends on N only through its parity $(-1)^N$.

[2] The requirement of including the special case (1.2), which corresponds to $\alpha_- = -\alpha_+$, $\beta_- = -\beta_+$, $k = 0$, helps to resolve the sign ambiguity in passing from (3.22) to (3.24).

We make the ansatz $Q_k = Q(u + \eta k)$, where $Q(u)$ is given by

$$Q(u) = \prod_{j=1}^{M} \sinh(u - u_j) \sinh(u + u_j + \eta) \tag{3.28}$$

which has the crossing symmetry $Q(u) = Q(-u - \eta)$. The zeros u_j of $Q(u)$ are still to be determined. Equations (3.27) and (3.14) imply that the eigenvalues are given by

$$\Lambda(u) = h(u)\frac{Q(u-\eta)}{Q(u)} + h(-u-\eta)\frac{Q(u+\eta)}{Q(u)}. \tag{3.29}$$

We verify that this result is consistent with both the periodicity (2.10) and crossing (2.11) properties of the transfer matrix. The requirement that $\Lambda(u)$ be analytic at $u = u_j$ yields the Bethe ansatz equations

$$\frac{h(u_j)}{h(-u_j-\eta)} = -\frac{Q(u_j+\eta)}{Q(u_j-\eta)} \qquad j = 1, \ldots, M. \tag{3.30}$$

The asymptotic behaviour (2.12), together with result (3.29) for the eigenvalues and constraint (3.25), implies that the number M of Bethe roots is given by

$$M = \tfrac{1}{2}(N - 1 + k) \tag{3.31}$$

where k is the integer appearing in (3.25). We leave to a future investigation the interesting question of determining further restrictions on the value of k, which presumably is related to the question of completeness.

We now argue that (3.21) is essentially the unique solution of (3.19). Indeed, if $\tilde{g}_{\mp}(u)$ are also solutions of (3.19), then

$$\tilde{g}_{-}(u)\tilde{g}_{+}(u) = g_{-}(u)g_{+}(u)\phi(u) \tag{3.32}$$

where $g_{\mp}(u)$ are given by (3.21), and $\phi(u)$ satisfies

$$\phi(u)\phi(-u) = 1. \tag{3.33}$$

The periodicity condition (3.15) implies that $\phi(u)$ has the same periodicity

$$\phi(u + \mathrm{i}\pi) = \phi(u). \tag{3.34}$$

We infer from (3.33) and (3.34) that $\phi(u)$ is a CDD-like factor

$$\phi(u) = \prod_j \frac{\sinh(u + v_j)}{\sinh(u - v_j)}. \tag{3.35}$$

The requirement that $\Lambda(u)$ be analytic then restricts $\phi(u)$ to the form

$$\phi(u) = \frac{q(u-\eta)}{q(u)} \qquad \text{where} \quad q(u) = \prod_j \sinh(u - v_j)\sinh(u + \eta + v_j). \tag{3.36}$$

This is equivalent to having additional Bethe roots, which can be included in $Q(u)$ (3.28).

Although the above results for the eigenvalues (3.26), (3.29), (3.30) have been obtained under the assumption that η is restricted to the values (3.6) with p odd, we expect that these results remain valid for generic values of η. Indeed, we have explicitly verified that these expressions reproduce the correct eigenvalues for $N = 0$ (with $M = 0$) and $N = 1$ (with $M = 1$) for arbitrary η.

4. Conclusion

Our proposed expression for the eigenvalues $\Lambda(u)$ of the transfer matrix (2.1) corresponding to the Hamiltonian (1.3) is given by (3.29), where $h(u)$ and $Q(u)$ are given by (3.26) and (3.28), respectively; the Bethe ansatz equations are given by (3.30), with M given by (3.31); and the parameters $\alpha_{\mp}$, $\beta_{\mp}$ (which are related to $\xi_{\mp}$, $\kappa_{\mp}$ by (3.24)) must satisfy the constraint (3.25).

It remains an open question whether a solution with Bethe ansatz equations of the 'conventional' form (3.30) can be found which does not require a constraint among the boundary parameters. (Although the solution proposed in [8] does not require any constraint among the boundary parameters, it holds only for the η values (3.6), and the Bethe ansatz equations are not of the conventional form.)

For the special case (1.2), an analysis of the thermodynamic ($N \to \infty$) limit and an extension to higher-dimensional representations have recently been given in [16]. For the more general case discussed here, it should now be possible to address such questions, and also to find generalizations to higher rank algebras, for both the trigonometric and elliptic cases.

Acknowledgments

I am grateful to all the organizers of the Annecy workshop 'Recent Advances in the Theory of Quantum Integrable Systems' for the opportunity to present this work. I am also grateful to many of the participants, in particular A Belavin, A Doikou, B McCoy, F Ravanini, V Rittenberg and S Ruijsenaars, for their questions or comments. This work was supported in part by the National Science Foundation under grant PHY-0098088.

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